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Inscribe in a circle any triangle ABC with the angle at B equal to 3β (Fig. 2). Draw AM, the diameter of the circle. Arc AH = arc HK = arc KC. Draw the chords AK, HC, KC, HM, KM, BH, and BK.

Hence,

$$\sin 3\beta = \frac{AC}{2r}.$$

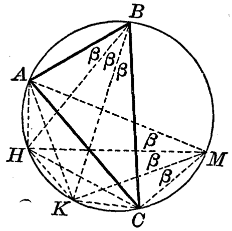


FIG. 2.

$AHKC$ is a concylic quadrilateral. Hence,

$$AC\cdot HK + AH\cdot KC = AK\cdot HC,$$

or

$$AC\cdot AH + \overline{AH}^2 = AK\cdot HC = \overline{AK}^2;$$

whence,

$$AC = (\overline{AK}^2 - \overline{AH}^2)/AH.$$

Hence,

$$\sin 3\beta = (\overline{AK}^2 - \overline{AH}^2)/2r\cdot AH.$$

But

$$AK = HC = 2r \sin 2\beta = 4r \sin\beta \cos\beta,$$

and

$$AH = 2r \sin\beta.$$

Hence,

$$\sin 3\beta = \frac{16r^2 \sin^2\beta \cos^2\beta - 4r^2 \sin^2\beta}{4r^2 \sin\beta} = 3 \sin\beta - 4 \sin^3\beta.$$

(4) $\cos 3\beta = 4 \cos^3\beta - 3 \cos\beta$.

From figure 2, we have

$$\cos 3\beta = \frac{MC}{2r}.$$

Also, since $MC^2 = AM^2 - AC^2$,

$$\frac{\overline{MC}^2}{4r^2} = \frac{4r^2 - \overline{AC}^2}{4r^2}.$$

But

$$\overline{AC}^2 = \frac{\overline{AK}^4 - 2\overline{AK}^2\cdot\overline{AH}^2 + \overline{AH}^4}{\overline{AH}^2}.$$

Hence,

$$\frac{\overline{MC}^2}{4r^2} = 1 - \frac{\overline{AK}^4}{4r^2\overline{AH}^2} + 2\frac{\overline{AK}^2}{4r^2} - \frac{\overline{AH}^2}{4r^2}.$$

Substituting the values of AK and AH, we have

$$\frac{\overline{MC}^2}{4r^2} = 1 - 16 \sin^2\beta \cos^4\beta + 8 \sin^2\beta \cos^2\beta - \sin^2\beta = 16 \cos^6\beta - 24 \cos^4\beta + 9 \cos^2\beta,$$

or

$$\frac{MC}{2r} = 4 \cos^3\beta - 3 \cos\beta.$$

Hence,

$$\cos 3\beta = 4 \cos^3\beta - 3 \cos\beta.$$

Also solved by A. M. HARDING and J. VINCENT BALCH.

CALCULUS.

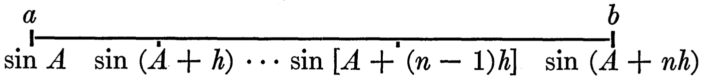
366. Proposed by I. A. BARNETT, University of Chicago.

Compute the definite integral $\int_a^b \sin^{-1} x dx$, where $0 \leqq a \leqq 1$, $0 \leqq b \leqq 1$, by direct summation.

SOLUTION BY THE PROPOSER.

Let $\sin A = a$, or $A = \sin^{-1} a$, and let $b = \sin(A + nh) = \sin B$. Hence $n = (B - A)/h$.

Then we have



$$\begin{aligned}\int_a^b \sin^{-1} x dx &= \lim_{h \doteq 0} [\{\sin(A + h) - \sin A\} \sin^{-1} \sin A \\ &\qquad + \{\sin(A + 2h) - \sin(A + h)\} \sin^{-1} \sin(A + h) + \cdots \\ &\qquad + \{\sin(A + nh) - \sin[A + (n-1)h]\} \sin^{-1} \sin[A + (n-1)h]] \\ &= \lim_{h \doteq 0} [- h\{\sin(A + h) + \sin(A + 2h) + \cdots + \sin(A + nh)\} - A \sin A \\ &\qquad + (A + nh) \sin(A + nh)] \\ &= \lim_{h \doteq 0} \left[- \frac{h/2}{\sin h/2} \left\{ \cos\left(A + \frac{h}{2}\right) - \cos\left[A + (2n+1)\frac{h}{2}\right] \right\} - A \sin A \right. \\ &\qquad \left. + (A + nh) \sin(A + nh) \right] \\ &= \lim_{h \doteq 0} \left[- \frac{h/2}{\sin h/2} \left\{ \cos\left(A + \frac{h}{2}\right) - \cos\left(B + \frac{h}{2}\right) \right\} - A \sin A + B \sin B \right] \\ &= \cos B - \cos A + B \sin B - A \sin A \\ &= \sqrt{1 - b^2} - \sqrt{1 - a^2} + b \sin^{-1} b - a \sin^{-1} a,\end{aligned}$$

which agrees with the result given in the tables.

An excellent solution was also received from A. M. HARDING.

367. Proposed by C. N. SCHMALL, New York City.

Show that the volume inclosed by the surface $(x^2 + y^2 + z^2)^5 = (a^3x^2 + b^3y^2 + c^3z^2)^2$ is $\frac{4}{9}\pi(a^3 + b^3 + c^3)$.

SOLUTION BY A. M. HARDING, University of Arkansas, AND A. R. NAUER, St. Louis, Mo.

Let $x = r \sin\varphi \cos\theta$, $y = r \sin\varphi \sin\theta$, $z = r \cos\varphi$. Substituting in the given equation, we obtain

$$r^3 = a^3 \sin^2\varphi \cos^2\theta + b^3 \sin^2\varphi \sin^2\theta + c^3 \cos^2\varphi.$$

Then the volume inclosed by the surface is given by

$$\begin{aligned}V &= 4 \int_0^{\pi/2} \int_0^{\pi} \int_0^{r} r^2 \sin\varphi dr d\theta d\varphi, \\ &= \frac{4}{3} \int_0^{\pi/2} \int_0^{\pi} (a^3 \sin^2\varphi \cos^2\theta + b^3 \sin^2\varphi \sin^2\theta + c^3 \cos^2\varphi) \sin\varphi d\theta d\varphi, \\ &= \frac{4}{3} \int_0^{\pi/2} \left(\frac{\pi}{2} \cdot a^3 \sin^3\varphi + \frac{\pi}{2} b^3 \sin^3\varphi + \pi c^3 \cos^2\varphi \sin\varphi \right) d\varphi, \\ &= \frac{4\pi}{3} \left[\frac{a^3}{2} \cdot \frac{2}{3} + \frac{b^3}{2} \cdot \frac{2}{3} + \frac{c^3}{3} \right] = \frac{4\pi}{9} (a^3 + b^3 + c^3).\end{aligned}$$